WHERE IS IT?
NEW TRACKING ALGORITHMS WITH REFRACTION ERROR

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1. OBJECTIVE

The objective is to provide a closed form expression relating the ray path and the index of refraction.

Tacking sensors such as lidar and radar can have appreciable errors in range and elevation due to refraction. Error inducing refraction effects extend up into the stratosphere but are generally larger near the ground. Thus, several models have been developed to estimate refraction error by using measured weather elements to calculate the refractive index and relating the refractive index to the path ray of a sensor.

An integral is required to calculate the ray path. The problem has been that no closed form solution has been found for this integral and numerical and iterative methods are required to calculate error due to refraction.

2. BACKGROUND

The refractive index of a medium is defined as ratio of the propagation velocity of electromagnetic energy in a vacuum to the propagation velocity in a medium. The refractive index is always greater than one. Air at the surface has a refractive index that can vary with weather conditions. Some typical surface values are from 1.00024 to 1.00035. The atmosphere approaches a vacuum with height; thus, the refractive index approaches one. Specific variations for a given location and altitude can be obtained from the REFRACT computer program produced by the Combat Climate Center, Ashville, NC.

The refractive index does depend on frequency. Both good approximations and more exact equations can be found in *The Handbook of Geophysics and the Space Environment* (Jursa, 1985) in equations 18.3 for visible and infrared frequencies and 19.7 for radio frequencies.

The reduced speed of propagation causes an error in calculating distance from the time of a returned radar or lidar signal. But of greater concern is bending due to variations of refractive index along the ray path. The amount of bending can be calculate using Snell's law (Moreland, 1965).

3. TIME AND SPACE SCALES

3.1 Scintillation

Scintillation or rapid changes in refraction occur at times less than one second. These are caused by small scale perturbations in the atmosphere. It is postulated that these are several causes of these perturbations. The most common is by "bubbles" of hot air rising similar to bubbles in boiling water. These “bubbles” are related to the updraft cells in cumulus clouds.

Another postulated cause is mechanical turbulence caused by the wind blowing over a rough surface. This effect is mitigated in that in such situations the air is well mixed and does not contain domains of deferent refractive index.

A third postulated cause is a ray passing through an uneven boundary between two layers. That is, the boundary is not flat but has gravity waves with troughs and crests and which occasionally even break much like waves on the sea. This type of small scale refractive index variation could be very dominant in stable regions of the atmosphere such as the stratosphere.

Optical systems such as lasers are much affected by scintillation but the effect on radar is minimal, not because of the different wave frequencies but because the radar beam is much wider and averages out much of the small scale refraction variations.

3.2 Spherically Stratified Layers

For time scales over one second or spatial scales large enough to average out scintillation effects, most of the variation in the refractive index occurs in the vertical. That is, the horizontal variation of the index of refraction is several orders of magnitude smaller then the variation in the vertical. Thus a model of horizontal homogeneity is often used. However, there may be situations where the horizontal variation is not negligible. For example, near the edge of a cumulonimbus (See figures 19-8 and 19-9, Jursa, 1985) or across the boundary of a strong cold front.

An appropriate form of Snell’s law for a vortically stratified continuously varying medium in spherical coordinates is,

$$nr\cos\theta = n_o r_o \cos\theta_o = K \quad \text{units L} \qquad [1]$$

where n is the index of refraction,

r is distance from the center of the earth,

θ is the elevation angle between the horizontal and the ray,

the o (origin) subscript refers to a starting point such as a radar antenna,

K is a constant determined by n_o, r_o, and θ_o.

3.3 Climatological scales

Since rays usually bend toward the earth, the distance a line-of-sight system can detect a target is further than straight geometric calculations would indicate. But because refraction varies, the effective distance of a system varies. An important climatological question often is how frequently will a system not work at a given distance and altitude.

4. RAY PATHS

From simple calculus, the relation between the elevation angle, θ, the height of a point on the ray measured from the center of the earth, r, and the angle from the center of the earth between the origin and a point on the ray, φ, is given by,

$$\cot\theta = \frac{r\,d\phi}{dr} \quad \text{no units} \qquad [2].$$

Combing [1] and [2] gives,

$$\left[\frac{dr}{r\,d\phi}\right]^2 = \left[\frac{nr}{K}\right]^2 - 1 \quad \text{no units} \qquad [3].$$

Solving for n gives,

$$n^2 = \left[\frac{K}{r}\right]^2 \left(\left[\frac{dr}{r\,d\phi}\right]^2 + 1\right) \quad \text{no units} \qquad [4].$$

Since in most problems, n is known as function of r, one would expect that [3] or [4] could be easily integrated at least layer by layer. However, if n is given the simple form where a, b, and c are fitted constants,

$$n = a + br \quad \text{no units} \qquad [5]$$

or the more realistic form,

$$n = a + \exp(bn + c) \quad \text{no units} \qquad [6],$$

no closed form integral appears to exist. Thus, numerical integration has been used. See for example, equations 71 to 75 in Moreland (1965). Modern computers handle these numerical summations quite well. However, they can be time consuming in real time operations or in real time hardware in-the-loop testing. Particularly in the simulation scenario where the location of the target is given and iterative methods are required to find θ_o. It is this type of problem that motivated a search for a closed form solution.

5. CLOSED FORM SOLUTIONS

Equations [3] or [4] can be separated into dφ and dr terms,

$$d\phi = \frac{K}{r^2\sqrt{n^2 - K^2/r^2}}\,dr \quad \text{no units} \qquad [7]$$

A form for n as a function of r can be hypothesize, put into [7], and then an attempt can be made to integrate the right hand side of [7] with respect to r.

5.1 A Linear Approximation

As mentioned above, linear and exponential forms do not lead to a closed form solution. However, other models of n as a function of r do! In particular, one useful form is,

$$n = a + b/r \qquad \text{no units} \qquad [8]$$

where a and b are constants for a given ray and can be calculated,

$$b = \frac{n_o - n_1}{1/r_o - 1/r_1} \qquad \text{units L} \qquad [9]$$

$$a = n_o - b/r_o \qquad \text{no units} \qquad [10]$$

where n_1 and r_1 are at some arbitrary level. Indeed, the Combat Climatology Center REFRACT program provides climatological values for the gradient of n from which b and a can be calculated.

The closed form solution for the ray is then,

$$\phi = c\left[\sin^{-1}(d + e/r) - f\right] \qquad \text{no units (radians)} \qquad [11]$$

where c, d, e, and f are constants for a ray and simple functions of K, a, and b,

$$c = \frac{K}{\sqrt{K^2 - b^2}} \qquad \text{no units} \qquad [12]$$

$$d = \frac{b}{K} \qquad \text{no units} \qquad [13]$$

$$e = \frac{b^2 - K^2}{aK} \qquad \text{units L} \qquad [14]$$

$$f = \sin^{-1}(d + e / r_o) \qquad \text{no units (radians)} \qquad [15]$$

Let us define the effective top of the atmosphere as the attitude that the difference in n is negligible from a vacuum. Since r is measured from the center of the earth, the variation in r from the surface to the effective top of the atmosphere is relatively small. For such a small variation in r the term b/r is nearly a straight line. Thus, the model, n=a + b/r can be used when linearly interpolating between know or measured values of n such as would be available from a radiosonde.

5.2 More General Forms for n

Writing the right hand side of equation [7] in the form,

$$\frac{dr}{\sqrt{\frac{n^2 r^4}{K^2} - r^2}} \qquad \text{no units} \qquad [16]$$

and if F(r) is a general function of r, the form of n,

$$n^2 = \frac{K^2}{r^2} + \frac{F(r)^2}{r^2} \qquad \text{no units} \qquad [17]$$

leads to relatively simple integral,

$$\int \frac{dr}{F(r)} \qquad \text{no units} \qquad [18]$$

from which many closed form solutions can be found. Recall that $1/r^2$ because r is the distance from the center of the earth, varies nearly linearly.

6. Conclusions

The closed form solutions for the ray path simplify ray tracing applications. Direct error analysis is possible. In ducts, the trapping angle can be directly calculated. When programing ray trace applications there is less chance of blunder.

7. REFERENCES

Jursa, Adolf, S., 1985: *Handbook of Geophysics And The Space Environment,* National Technical Information Service, ADA 167000.

Moreland, William. B., 1965: *Estimating Meteorological Effects On Radar Propagation,* Technical Report 183, volume 1, Air Weather Service.